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Flag Spaces in KP Theory and Virasoro Action on $\det D_j$ and Segal-Wilson τ-Function[1]

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Abstract.
We consider Virasoro action on flag spaces corresponding to the Riemann surfaces with two marked points.

0. Introduction. The development of string theory and conformal theories on Riemann surfaces has produced interest in the objects of soliton theory (see, for example, [1]). There are a number of papers using the Segal-Wilson Grassmannians as a model of universal moduli space – the space containing all the Riemann surfaces of finite genus. In the case of superstrings, which appears to be simpler, the measure was calculated in [2]. Another soliton object – the τ-function introduced by the Kyoto mathematicians (see [29], [3] and references therein) – may be defined as some vacuum expectation of fermionic fields [4]. The monodromy properties of the τ-function [5] were used to calculate the $\det \bar\partial$ for hyperelliptic curves [6].

The Segal-Wilson Grassmannians correspond to Riemann surfaces with a single marked point. However it is more natural to consider Riemann surfaces with a number of marked points. In the simplest case we have two

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marked points which correspond to in and out states of the string. Conformal field theory on such surfaces was constructed by I.M.Krichever and S.P.Novikov [7]-[9], where they introduced some analogue of the Laurent basis for tensor fields on such surfaces. We show that the proper analogue of the Grassmannian in this situation is the flag space.

In conformal theories the following algebras play a crucial role – the algebra of the vector fields on the circle and its central extension known as the Virasoro algebra. In the Krichever-Novikov basis the Virasoro algebra appears to be generalized-graded.

In our paper we consider the following object: a Riemann surface Γ, with two marked points P_0 and ∞ and divisor $\gamma_1, \ldots, \gamma_g$. The points P_0 and ∞ play different roles in our approach. We consider a small contour S around ∞, and the space $\mathcal{L}^2(S)$ of a set of elements $W(t_0), t_0 \in \mathbb{Z}$, where $W(t_0)$ consists of functions meromorphic outside ∞ with divisor $t_0 P_0 - \gamma_1 - \ldots - \gamma_g, W(t_0+1) \subset W(t_0), W = \{W(t_0)\}$ is an element of the flag space. If γ_i are located in the points P_0 and ∞, then $W(t_0)$ are generated by the elements of the Krichever-Novikov basis. We also fix a local parameter z in ∞. The Krichever construction (see review, [16]) allows us to construct solutions of the Kadomtsev-Petviashvili equation via such data depending on an extra parameter t_0. In the KP theory, t_0 was introduced in [3],[28], where t_0 was treated as a discrete time in the generalized hierarchy containing KP and Toda lattice hierarchies. In string theory, t_0 plays the role of momentum.

In our paper we study the Virasoro action on the KP theory objects generated by the Virasoro action on the Riemann surfaces. Our main tool is the Cauchy-Baker-Akhiezer kernel (see section 2.4), which inverts the $\bar{\partial}$ operator on a certain bundle, $\mathbb{B}_j(t_0, \vec{t}, D)$. With the help of this kernel we present an explicit version of the Segal-Wilson construction. We show that the Virasoro action on the KP potentials coincides with the non-isospectral KP-hierarchy [10]. The times of equations from this hierarchy correspond to the deformations of the Riemann surfaces and form the coordinates on the moduli space. We also calculate the action on the Baker-Akhiezer function, i.e., solve the problem by I.M.Krichever and S.P.Novikov. In these cases we have no central extension. Then we introduce the τ-function corresponding to the j-tensors, and calculate the Virasoro action on it. This action is represented by second order differential operators acting on the space of functions of an infinite number of variables. The central charge is $c_j = 6j^2 - 6j + 1$ in accordance with [11]. (The Virasoro action on the j-tensors Grassmanniansand

central extensions were considered in [2]). This representation is valid for arbitrary τ-functions. In the case of the algebro-geometrical τ-function, we have θ-functional realizations of generalized Verma modules in the sense of [7]-[9], parameter t_0 playing the role of highest weight. "Naive" calculation of the variation of the $\det \bar{\partial}_j$ in the corresponding bundle gives the same results as the calculation of the τ-function variation. Nontrivial bundles are necessary to suppress $(2j-1)(g-1)$ zero modes of operator $\bar{\partial}_j$. We use the bundle corresponding to the Baker-Akhiezer functions. So we may treat the Segal-Wilson τ-function as $\det \bar{\partial}_j$. We discuss the connection between the τ-function and the Krichever-Novikov vacuum expectation $\mathcal{A}$ [8],[9].

Chapter 1. Riemann surfaces and bundles.

1.1. *$\bar{\partial}_j$-operator and index. The Cauchy kernel.* Consider the following equation on the Riemann surface Γ of genus g:

$$\bar{\partial} f = \varphi \tag{1.1}$$

in the simplest case, when f is a function and φ is a $(0,1)$-tensor i.e. $\varphi = \tilde{\varphi}(z,\bar{z})d\bar{z}$. The difference between the dimensions of Ker $\bar{\partial}$ and Coker $\bar{\partial}$ is called the index of $\bar{\partial}$. Assume that $\bar{\partial}$ maps the nonsingular functions to the nonsingular forms. In this case $\bar{\partial}$ has one-dimensional kernel ($\bar{\partial}\cdot \mathrm{const} = 0$) and a g-dimensional cokernel – that is, index $\bar{\partial} = 1 - g$. For (1.1), the g-dimensional cokernel means that (1.1) has a solution if and only if φ satisfies g linear relations:

$$\int\int_\Gamma \tilde{\varphi}(z,\bar{z})\tilde{w}_n(z)dzd\bar{z} = 0,\ n = 1, 2, \ldots, g, \tag{1.2}$$

where $\tilde{w}_n(z)dz$ are the holomorphic differentials (see (1.4)).

In the Quillen theory of $\det \bar{\partial}$ [26], index $\bar{\partial}$ is assumed to be zero. We can make $\bar{\partial}$ be of index zero and invertible by assuming f and φ to be elements of the following nontrivial bundles. Let $\gamma_1, \ldots, \gamma_g, P$ be a collection of points of general position. The functions f and φ are smooth everywhere, except for singularities of the simple poles type $\varphi_k(z,\bar{z})/(z-\gamma_k)$ in γ_k with smooth φ_k. f and φ have simple zeroes in P, of the form $d(z,\bar{z})(z-P)$. Then (1.1) has a unique solution:

$$f(z,\bar{z}) = \int\int \tilde{\omega}(z,z')\tilde{\varphi}(z',\bar{z}')dz'd\bar{z}', \tag{1.3}$$

where the Green function $\omega = \tilde{\omega}(z,z')dz'$ has the following properties:

1) ω is a meromorphic 0-form in z and a meromorphic 1-form in z';

2) ω has simple poles (zeroes) in $\gamma_1,\dots,\gamma_g$ and a simple zero (pole) in P as a function of z (of z'), respectively;

3) $\omega \simeq (2\pi i)^{-1}dz'/(z'-z)$ as $z \to z'$.

ω is the meromorphic analogue of the Cauchy kernel on the Riemann surface [13].

The functions f and φ can be interpreted as smooth sections of nontrivial holomorphic bundles.

Remark. Even in the simplest case of $g=0$ and the ordinary Cauchy kernel, bundles are nontrivial and correspond to a simple zero in ∞.

1.2. *Notation.* In our paper we assume that the Riemann surfaces Γ are compact. Let g be the genus of Γ, $g<\infty$. We will need the following geometrical objects on Γ [14],[15],[16]:

1) $\vec{\mathbf{w}} = (\mathbf{w}_1,\dots,\mathbf{w}_g)$ - the basis of holomorphic 1-differentials with the following standard normalization

$$\oint_{a_i} \mathbf{w}_k = \delta_{ik}, \oint_{b_i} \mathbf{w}_k = B_{ik}, \tag{1.4}$$

B_{ik} is called the matrix of periods or Riemann matrix.

2) The Abel transformation. Let P be a collection of points $P=(P_1,\dots,P_n)$. Then

$$\vec{A}(P) = \int_\infty^{P_1} \vec{\mathbf{w}} + \int_\infty^{P_2} \vec{\mathbf{w}} + \ldots + \int_\infty^{P_n} \vec{\mathbf{w}}. \tag{1.5}$$

3) The prime-form $E(\gamma,\gamma'), \gamma,\gamma' \in \Gamma$ which is a holomorphic $-1/2$-form in γ and in γ' with the following properties:

a) $E(\gamma,\gamma')=0$ if and only if $\gamma=\gamma'$.

b) Let t be a local coordinate on Γ. Then for $\gamma\to\gamma'$:

$$E(\gamma,\gamma') = \frac{(t(\gamma)-t(\gamma'))\{1+O((t(\gamma)-t(\gamma'))^2\}}{\sqrt{dt(\gamma)dt(\gamma')}}. \tag{1.6}$$

c) $E(\gamma,\gamma')$ is a multivalued form in Γ with the following periodic conditions: $E(\gamma+a_k,\gamma') = E(\gamma,\gamma')$, $E(\gamma+b_k,\gamma') = \pm E(\gamma,\gamma')\exp(-\pi i B_{kk} + 2\pi i \int_\gamma^{\gamma'} \mathbf{w}_k)$ where a_k and b_k are the basic cycles.

4) Meromorphic differentials Ω_k and dp_k. Let us have a fixed point ∞ in Γ with a local parameter z. Then we introduce differentials Ω_k and dp_k with

the unique pole at ∞ such that $\Omega_k = d(1/z^k) + O(1)$, $idp_k = d(1/z^k) + O(1)$ and

$$\oint_{a_l} \Omega_k = 0,\ \mathrm{Im} \oint_{a_l} dp_k = 0,\ \mathrm{Im} \oint_{b_l} dp_k = 0,\ l = 1, \dots g. \tag{1.7}$$

The multivalued functions p_k are called quasimomentums [16]. In the soliton theory they correspond to the times t_k. The functions $\mathrm{Im}\, p_k$ are correctly defined on Γ. For Ω_k we have:

$$d_\gamma d_z \ln E(\gamma, z) = -\sum_1^\infty \Omega_k(\gamma) z^{k-1} dz. \tag{1.8}$$

5) In the neighbourhood of ∞ we have the following expansions:

$$\ln \frac{E(z,z')}{z-z'} = \sum_{m\geq 2} \frac{Q_{m0}(z^m + (z')^m)}{m} + \sum_{m,n\geq 1} \frac{Q_{mn} z^m (z')^n}{mn},\ Q_{mn} = Q_{nm}, \tag{1.9}$$

$$\Omega_k = d\left(\frac{1}{z^k}\right) - \sum_{m\geq 1} Q_{km} z^{m-1} dz,\ k \geq 1, \tag{1.10}$$

$$\vec{\mathbf{w}} = -\sum_{k\geq 1} \vec{U}_k z^{k-1} dz,\ \text{where } (\vec{U}_k)_m = (2\pi i)^{-1} \oint_{b_m} \Omega_k. \tag{1.11}$$

1.3. *Divisors and holomorphic bundles on the Riemann surfaces.* Let us recall some necessary constructions from algebraic geometry. A divisor is a formal linear combination of points of the Riemann surface Γ: $D = \sum n_i \gamma_i$. If f is a meromorphic function on Γ with poles γ_i of order m_i and zeroes $\gamma_i^*$of order n_i, then the divisor $D(f) = \sum -m_i\gamma_i + n_i\gamma_i^*$ corresponds to it. Two divisors D and D' are called equivalent if $D - D'$ is a divisor of some meromorphic function. The classes of equivalent divisors form the Picard group $\mathrm{Pic}\,(\Gamma)$. Let $D_1 = \sum n_i\gamma_i$, $D_2 = \sum \tilde{n}_i\gamma_i$ (some of $n_i, \tilde{n}_i$ can be equal to 0). Then $D_1 \geq D_2$ if $n_i \geq m_i$ for all i. The sum $\deg(D) = \sum -m_i + n_i$ is called the degree of D.

A holomorphic bundle is a bundle with a holomorphic gluing law. It is very convenient to describe one-dimensional holomorphic bundles via divisors. Let $\mathbb{B}$ be a one-dimensional holomorphic bundle, $s(\gamma)$ be its global meromorphic section, $D(s)$ be the divisor of s, and b be the element of $\mathrm{Pic}\,(\Gamma)$ generated by $D(s)$. The divisor $D(s)$ depends of course on the section $s(\gamma)$; but different sections result in equivalent divisors, so the map

$\mathbb{B} \to b \in$ Pic$\,(\Gamma)$ is correctly defined. In algebraic geometry the following statement is well-known:

Lemma 1.1. The map from the set of one-dimensional holomorphic bundles on Γ to the Pic$\,(\Gamma)$ group is a one to one correspondence.

Let $\mathbb{B}$ be a holomorphic bundle on Γ with a global meromorphic section s – the so-called equipped bundle; U be a domain in Γ; and $D'(s)$ be the restriction of $D(s)$ on U. Then the holomorphic sections t of $\mathbb{B}$ on U can be represented by the meromorphic functions f in U such that $D(f)+D'(s) \geq 0$ in the following way: $t = f{\cdot}$s. The meromorphic sections with divisor $D(t) \geq D_0$ correspond to meromorphic functions such that $D(f) + D'(s) \geq D_0$. Thus we can speak about meromorphic functions with prescribed singularities instead of holomorphic sections of bundles.

If the section $s(\gamma)$ has no zeroes and poles in U, we have a trivialization of $\mathbb{B}$ on U and $s(\gamma)$ is called a unit section.

We shall also speak about meromorphic j-tensors with a given set of zeroes and poles. Such bundles are isomorphic to bundles of 0-forms with different divisors. But we shall not use this isomorphism because we need to vary the basic curve Γ. If we vary the basic curve it is necessary to describe how the bundles are varied and this variation will depend on the tensor weight j.

The multidimensional bundles are not considered here.

1.4.*Deformations of Riemann surfaces and the Riemann problem.* In this section we consider how the algebra of the vector fields on the circle varies the structures of Riemann surfaces [17]. Let S be a small circle around ∞ on Γ and $U(S)$ be its small neighbourhood such that $\infty \in U(S)$. Let Γ be covered by two regions Γ_+ and Γ_- such that $U(S) = \Gamma_+ \cap \Gamma_-$ and $\infty \in \Gamma_-$. Γ may by treated as a result of gluing Γ_+ and Γ_-. We may vary the Riemann surface Γ by changing the gluing law. Let us describe this change. Let $v = \tilde{v}(z)d/dz$ be a holomorphic vector field in the region $U(S)$, and $\exp(\beta v)\gamma_-$ be the shift of the point γ_- along v after the lapse of time β. Let the original gluing law be $\gamma_+ \to \gamma_-$, $\gamma_- \in \Gamma_-$, $\gamma_+ \in \Gamma_+$. Now we obtain a new Riemann surface $\Gamma^{'}$ by gluing the point γ_+ to the point $\exp(\beta v)\gamma_-$. Both Riemann surfaces Γ and $\Gamma^{'}$ are constructed of the same regions Γ_+ and Γ_-. Then the unit maps $\Gamma_+ \to \Gamma_+$, $\Gamma_- \to \Gamma_-$ define a natural mapping $\Gamma \to \Gamma^{'}$ with a jump on S. We call this mapping E. When calculating the commutators of vector field actions we assume all vector fields to be independent of β functions of local parameter $z = 1/\lambda, z$ to be defined on Γ_-, and when we vary Riemann

surfaces we map z by E.

Let D_0 be a divisor on Γ and $\mathbb{B}_j$ be the bundle of j-tensors f_j such that $D(f_j) \geq -D_0$. We assume that the corresponding bundle $\mathbb{B}_j^{'}$ on the new surface $\Gamma^{'}$ is the bundle of j-tensors $f_j^{'}$ such that $D(f_j^{'}) \geq -E(D_0)$.

In the case of infinitesimal action ($\beta << 1$), a holomorphic j-tensor field Δ' on the new surface $\Gamma^{'}$ can be treated as a field on the old surface Γ with a jump on S satisfying the following equation

$$\Delta'_{+} - \Delta'_{-} = \beta L_v \Delta \tag{1.12}$$

where Δ'_{+} and Δ'_{-} are the boundary values of Δ' on S, L_v is the Lie derivative, and Δ is the original field on Γ. Thus Δ' is a solution of the Riemann problem – a well-known object of soliton theory. We assume in our paper that the index of (1.12) is zero. The Riemann problem (1.12) can be considered as a special case of the $\bar{\partial}$-problem (1.1) with a δ-type function g. Then it's solution is given by:

$$\Delta'(\gamma) = \Delta(\gamma) + \oint_S \omega(\gamma, \gamma')\beta L_v \Delta(\gamma'), \tag{1.13}$$

where $\omega(\gamma, \gamma')$ is the same Cauchy kernel as in the $\bar{\partial}$-problem (the kernel $\omega(\gamma, \gamma')$ depends, of course, on the bundle $\mathbb{B}_j$). For example a calculation for a holomorphic 1-form using (1.13) gives rise to the well-known formula for the variation of Riemann matrix B_{mn} [17]:

$$\frac{\partial B_{mn}}{\partial \beta} = \oint_S \tilde{v}(z)\tilde{\mathbf{w}}_m(z)\tilde{\mathbf{w}}_n(z)dz, \tag{1.14}$$

where $\tilde{\mathbf{w}}_m(z)dz$, $m = 1, \ldots, g$ is the basis of holomorphic 1-forms.

Chapter 2. Elements of the Kadomtsev-Petviashvili equation theory.

In this chapter the necessary definitions are introduced. The most important of them are the Baker-Akhiezer functions, the Baker-Akhiezer j-tensors and the Segal-Wilson τ-function. The main tool is the Cauchy-Baker-Akhiezer kernel which inverts the $\bar{\partial}$ operator. The first section illustrates the further considerations and may be omitted.

2.1 *Riemann surfaces in the integrable equations theory.* Now we shall show how the Riemann surfaces and the Baker-Akhiezer function appear in

the theory of nonlinear equations [18]. Then we shall show that the Virasoro action corresponds to higher symmetries of these equations. Consider the simplest example – the Korteveg-de Vries equation (KdV):

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.1}$$

It is a Hamiltonian equation

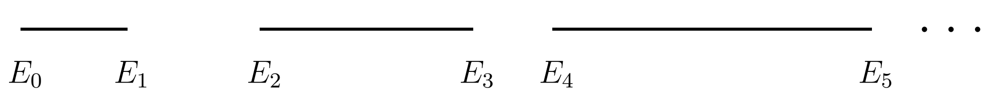
$$u_t = \frac{d}{dx}\frac{\delta H}{\delta u}, \ H = \int \left(\frac{1}{2}u_x^2 + u^3\right) dx \tag{2.2}$$

with an infinite set $H_1, H_2, \ldots$ of first integrals in involution, $H_n = \int h_n(u, u_x, \ldots)dx$.

The scheme of solving this equation is based upon the following representation for KdV (Lax representation). Let $L = -\partial^2/\partial x^2 + u(x,t)$, $A = \partial/\partial t + 4\partial^3/\partial x^3 - 6u\partial/\partial x - 3u_x$. Then operators L and A commute

$$LA = AL \tag{2.3}$$

if and only if $u(x,t)$ is a solution of (2.1). One can see from (2.3) that the spectrum of L is independent of t. (If we speak about the spectral properties of L, we consider it as an ordinary differential operator depending on a parameter t). If $u(x,t)$ is periodic in x: $u(x+T,t) = u(x,t)$ then the spectrum of L consists of a set of intervals $[E_0, E_1]$, $[E_2, E_3]$, $[E_4, E_5], \ldots$ $E_0 < E_1 < E_2 < \ldots$



The open intervals $(-\infty, E_0)$, $(E_1, E_2), \ldots$ are called gaps, $E_{2n} - E_{2n-1} \to 0$ as $n \to \infty$. The Bloch eigenfunction of the operator L (it coincides with the Baker-Akhiezer function in this case) is the solution of

$$L\psi(x,E,t) = E\psi(x,E,t) \tag{2.4}$$

such that $\psi(x+T,E) = \exp(iTp(E))\psi(x,E,t)$ with the normalization $\psi(0,E,t) = 1$. The function $\psi(x,E,t)$ is meromorphic in E on a Riemann surface Γ which is two-sheeted over the E-plane; the branch points are E_0, $E_1, \ldots$. The function $\psi(x,E)$ has exactly one pole $\gamma_n(t)$ and one zero $\gamma_n^+(x,t)$ over each gap except $(-\infty, E_0)$; $\gamma_n^+ = \gamma_n$ as $x = 0$.

If we know E_n and $\gamma_n(t)$, then the function $u(x,t)$ can be reconstructed in all x. The shift along the flow corresponding to the Hamiltonian H as well as H_m results in change in the positions of the poles γ_n, the points E_n being invariant. Finite-dimensional invariant subspaces correspond to the so-called finite-gap potentials, i.e., to the potentials such that the spectrum has the form $[E_0, E_1]$, ..., $[E_{2N}, \infty]$. These potentials are the stationary points of equations with Hamiltonians of the form $\mathbf{H} = \sum_{m=1}^{N} c_m H_m$, $\frac{d}{dx}\frac{\delta \mathbf{H}}{\delta u} = 0$. The restriction of the KdV equation on this subspace gives rise to finite-dimensional systems that are integrable in the Liouville sense. Roughly speaking, the action variables correspond to the sizes of gaps and the angle variables correspond to the positions of γ on the Riemann surface Γ.

Except for the symmetries corresponding to the Hamiltonians H_m, other flows commuting with KdV equation do exist [19]. They can be written in the following form:

$$\frac{\partial u}{\partial \beta_m} = \frac{d}{dx}\Lambda^{m+1} \cdot (6tu + x), \tag{2.5}$$

where

$$\Lambda = -\left(\frac{d}{dx}\right)^2 + 4\left(\frac{d}{dx}\right)^{-1} u \left(\frac{d}{dx}\right) + 2\left(\frac{d}{dx}\right)^{-1} u_x \tag{2.6}$$

is called a recursion operator [21].

Ordinary symmetries of the KdV equation which correspond to the Hamiltonians H_m, also can be written via Λ :

$$\frac{\partial u}{\partial t_m} = \frac{d}{dx}\Lambda^m \cdot \frac{1}{2}. \tag{2.7}$$

In our paper we study how equations (2.5) act on the finite-gap KdV solutions. The ends of gaps are not invariant under these flows, but:

$$\frac{\partial E_k}{\partial \beta_m} = E_k^{m+1}. \tag{2.8}$$

In particular flows $\partial E_k/\partial\beta_{-1} = 1$, $\partial E_k/\partial\beta_0 = E_k$, $\partial E_k/\partial\beta_1 = E_k^2$ which represent infinitesimal fractional transformations of the E-plane, correspond to:

$$\partial u/\partial\beta_{-1} = 6tu_x + 1 \qquad \text{(Galilean transformation)},$$

$$\partial u/\partial\beta_0 = 6tu_t + 2xu_x + 4u \qquad \text{(Scaling transformation)},$$

$$\partial u/\partial \beta_1 = 6t(u_{xxxx} - 10uu_{xx} - 5u_x^2 + 10u^3)_x + 2xu_t + 16u^2 + 4u_x \left(\frac{d}{dx}\right)^{-1} u.$$

The flows (2.5) commute as the corresponding vector fields $E^{m+1}\partial/\partial E$ (see (2.8)).

The shift (2.8) of the branch points E_k of the surface Γ generates the variation of the complex structure of Γ. As one can see, vector fields which do not move E_k appear to be generalized-graded [7].

In the KdV theory only the hyperelliptic Riemann surfaces emerge. Arbitrary Riemann surfaces appear in the theory of the KP equation:

$$(4u_t - u_{xxx} - 6uu_x)_x = 3u_{yy}. \tag{2.9}$$

The fact that the scaling transformation can be obtained from the Galilean one by applying the recursion operator was first pointed out in [20]. In [20] it was also shown that applying the recursion operator to the scaling symmetry we get a non-local KdV symmetry. But in [20] only the local symmetries were studied, thus the last observation had no consequences in the context of [20].

2.2 *The Krichever construction. The Baker-Akhiezer function.* (see review [16]). Let Γ be a Riemann surface of genus $g < \infty$ with a given point ∞, a local parameter $z = 1/\lambda$ in the neighbourhood of ∞, and a divisor $\gamma_1, \ldots, \gamma_g$. The Baker-Akhiezer function $\psi(\gamma, \vec{t})$ is a function uniquely determined by the following properties.

1) It depends on a spectral parameter $\gamma \in \Gamma$ and an infinite set of times $\vec{t} = (t_1, \ldots), t_1 = x, t_2 = y, t_3 = t$.

2) It is meromorphic by γ everywhere but ∞, and has simple poles in $\gamma_1, \ldots, \gamma_g$.

3) It has an essential singularity:

$$\psi(\gamma, \vec{t}) = (1 - \chi(\vec{t})/\lambda - o(1/\lambda)) \exp(\sum \lambda^m t_m), \ \lambda = \lambda(\gamma) \text{ as } \gamma \to \infty.$$

We call χ potential; it obeys the KP equation:

$$4\chi_{xt} = \chi_{xxxx} + 6\chi_x\chi_{xx} + 3\chi_{yy}.$$

(In (2.9) $u = \chi_x$.) The t_m- dependence of χ ($m > 3$) is described by the higher KP equations (the so-called KP-hierarchy). We need also a conjugated

Baker-Akhiezer differential $\psi^*(\gamma, t)$, which is holomorphic by γ 1-form on $\Gamma\backslash\infty$. $\psi^*(\gamma, t)$ has simple zeroes at $\gamma_1, \ldots, \gamma_g$ and an essential singularity

$$\psi^*(\lambda, \vec{t}) = (1 + O(1/\lambda)) \exp(-\sum \lambda^m t_m) d\lambda \text{ as } \lambda \to \infty.$$

(for explicit formulas for $\psi(\gamma, \vec{t})$ and $\psi^*(\gamma, \vec{t})$ see 2.3). We also need:

Lemma 2.1 (see [16]). Let $\operatorname{Im} p_1(\lambda) = \operatorname{Im} p_1(\mu)$ (for quasimomentum p_1, see 1.2). Then the functions ψ and ψ^* are orthogonal functions of x:

$$\int_{-\infty}^{+\infty} \psi(\lambda, x, y, t, \ldots)\psi^*(\mu, x, y, t, \ldots) dx = 0 \text{ as } \lambda \neq \mu.$$

2.3. *Baker-Akhiezer j-forms. The bundles* $\mathbb{B}_j(t_0, \vec{t}, D)$. Along with the Baker-Akhiezer functions we shall use Baker-Akhiezer j-differentials $\psi_j(\gamma, t_0, \vec{t})$, introduced in [7],[9] (Baker-Akhiezer 1-differentials were introduced in [22]).

Let $D = \sum n_m \gamma_m$ be some divisor of degree $(2j-1)(g-1)$ (see 1.3). We shall consider the following equipped holomorphic bundle $\mathbb{B}_j(t_0, \vec{t}, D)$: local holomorphic sections of $\mathbb{B}_j(t_0, \vec{t}, D)$ are meromorphic on $\Gamma\backslash\infty$ j-forms $f(\gamma)$ such that $D(f) \geq D + (t_0 - 1)P_0$ and $f(z)(z)^{t_0-1}(dz)^{-j} \exp - \sum z^{-m} t_m$ is regular at ∞ ($z = 1/\lambda$). The index of $\bar{\partial}_j$ on $\mathbb{B}_j(t_0, \vec{t}, D)$ equals 0. Thus we eliminate the $(2j-1)(g-1)$-dimensional kernel of $\bar{\partial}_j$ on the trivial bundle. For $\vec{t} = 0$, $t_0 = 1$ we denote $\mathbb{B}_j(t_0, \vec{t}, D) = \mathbb{B}_j$. The conjugate bundle to $\mathbb{B}_j(t_0, \vec{t}, D)$ is $\mathbb{B}^*_j(t_0, \vec{t}, D) = \mathbb{B}_{1-j}(2 - t_0, -\vec{t}, -D)$ (i.e, the bundle $\mathbb{B}^*_j(t_0, \vec{t}, D)$ consists of the $1-j$-differentials).

The Baker-Akhiezer j-differential $\psi_j(\gamma, t_0, \vec{t})$ and the conjugate $1-j$-differential $\psi^*_j(\gamma, t_0, \vec{t})$ can be defined as meromorphic sections of $\mathbb{B}_j(t_0, \vec{t}, D)$ and $\mathbb{B}^*_j(t_0, \vec{t}, D)$, respectively, with no singularities except simple poles in ∞. For $\psi_j(\gamma, t_0, \vec{t})$ we have the explicit formula:

$$\psi_j(\gamma, t_0, \vec{t}) = \eta(\gamma)\psi_{1/2}(\gamma, t_0, \vec{t}), \text{ where} \tag{2.10}$$

$$\eta(\gamma) = \left[\frac{\theta(\vec{A}(\gamma) - \vec{K})}{\theta^{(g)}(-\vec{K})E(\gamma, \infty)\sqrt{dz(\infty)}}\right]^{2j-1} \prod_k \left[\frac{E(\gamma, \gamma_k)}{E(\gamma, \infty)E(\infty, \gamma_k)dz(\infty)}\right]^{n_k},$$

$$\psi_{1/2}(\gamma, t_0, \vec{t}) = \left[\frac{E(\gamma, P_0)}{E(\gamma, \infty)E(\infty, P_0)dz(\infty)}\right]^{t_0-1} \frac{\theta(\vec{A}(\gamma) + \vec{\xi})}{\theta(\vec{\xi})E(\gamma, \infty)\sqrt{dz(\infty)}} \exp\left(\int^{\gamma} \sum_{k\geq 1} \Omega_k t_k\right),$$

$$\theta^{(g)}(-\vec{K}) = \frac{1}{g!}\frac{d^g}{dz^g}\,\theta(\vec{A}(z)-\vec{K})\Big|_{z=0},$$

$$\vec{\xi} = (2j-1)\vec{K} + \sum n_k \vec{A}(\gamma_k) + (t_0-1)\vec{U}_0 + \sum_{k\geq 1} \vec{t}_k \vec{U}_k + \vec{\zeta},$$

where $\vec{K}$ is the vector of Riemann constants; $\vec{U}_0 = \vec{A}(P_0)$: for Abel transform $\vec{A}$ see 1.2; $\vec{\zeta} = 0$; for θ-functions see [14]. The constants in the integrals in (2.10) are chosen so that $\int^\gamma \Omega_k = 1/z^k + o(1)$. When $\gamma \to \infty$, $\eta(\gamma) \sim (dz)^{j-1/2}$.

If $\vec{\zeta} \neq 0$ (2.10) results in Baker-Akhiezer functions with nonzero characteristics: $\psi_j(\gamma + a_k, t_0, \vec{t}) = \psi_j(\gamma, t_0, \vec{t})$, $\psi_j(\gamma + b_k, t_0, \vec{t}) = \psi_j(\gamma, t_0, \vec{t}) \cdot \exp(2\pi i \zeta_k)$ where a_k and b_k are basis cycles.

Not only integer j but $j \in \mathbb{Z}/2$ can be considered (see [9]).

Remark. For $j = 1/2$ we can take $D = 0$ and parameterize the Baker-Akhiezer functions by characteristics $\vec{\zeta}$.

The functions $\psi_j(\gamma, t_0, \vec{t})$ and $\psi^*_j(\gamma, t_0, \vec{t})$ have the following asymptotics, when $\gamma \to \infty$:

$$\psi_j(\gamma, t_0, \vec{t}) = z^{-t_0}(dz)^j \exp\left(\sum_{k\geq 1} z^{-k} t_k\right)(1 + O(z)),$$

$$\psi^*_j(\gamma, t_0, \vec{t}) = z^{t_0-2}(dz)^{1-j} \exp\left(-\sum_{k\geq 1} z^{-k} t_k\right)(1 + O(z)), \tag{2.11}$$

For $\gamma \to P_0$ we have:

$$\psi_j(\gamma, t_0, \vec{t}) \sim \varphi(t_0, \vec{t}) z^{t_0-1}(dz)^j, \quad \psi^*_j(\gamma, t_0, \vec{t}) \sim \varphi^*(t_0, \vec{t}) z^{1-t_0}(dz)^{1-j} \tag{2.12}$$

where z_- is some local parameter in P_0.

The variations of the Baker-Akhiezer forms are solutions of the corresponding Riemann problem in $\mathbb{B}_j(t_0, \vec{t}, D)$ (see 3.2).

For all j, the bilinear identity [3] $\oint_S \psi_j(\gamma, t_0, \vec{t}) \psi^*_j(\gamma, t'_0, \vec{t}') \equiv 0$ for $t_0 \geq t'_0$ holds. For $t_0 = t'_0$ it results in the ordinary KP-hierarchy. The corresponding solutions do not depend on j. However, for the equations corresponding to the changing complex structure, the tensor weight is important.

From (2.11) we have the orthogonality condition:

$$\oint_S \psi_j(\gamma, t_0, \vec{t}) \psi^*_j(\gamma, t'_0, \vec{t}) = -\delta(t_0 + 1, t'_0),$$

which was derived for $\vec{t}=0$ in [7]. Let us note that $\psi_j(t_0)$ and $\psi_j^*(t_0')$ form full mutually orthogonal bases (see [7]) as functions of γ.

Remark. If all points of γ_k coincide with P_0 or ∞ then for $\vec{t}=0$ we obtain the Krichever-Novikov bases for j-forms [7].

2.4. *"Cauchy-Baker-Akhiezer" kernel.* We shall solve the Riemann problem (1.12) and the $\bar{\partial}$-problem for the bundle $\mathbb{B}_j(t_0,\vec{t},D)$ with the help of the Cauchy-Baker-Akhiezer kernel:

$$\omega_j(\lambda,\mu,t_0,x,y,t,\ldots)=\int\limits_{\mp\infty}^{x}\psi_j(\lambda,t_0,x',y,t,\ldots)\psi_j^*(\mu,t_0,x',y,t,\ldots)\frac{dx'}{2\pi i}. \tag{2.13}$$

We choose the sign so that the integral converges. For $\operatorname{Im}p_1(\lambda)=\operatorname{Im}p_1(\mu)$ the definition (2.13) is correct because of the Lemma 2.1.

One can check that $\psi_j(\gamma,t_0+1,\vec{t})=(\partial/\partial t_1-v_j(t_0,\vec{t}))\psi_j(\gamma,t_0,\vec{t})$ and $\psi_j^*(\gamma,t_0,\vec{t})=(\partial/\partial t_1+v_j(t_0,\vec{t}))\psi_j^*(\gamma,t_0+1,\vec{t}),v_j(t_0,\vec{t})=\partial/\partial t_1\ln(\varphi(t_0,\vec{t}))$ (see 2.12). Therefore we have a different representation for the Cauchy-Baker-Akhiezer kernel:

$$\omega_j(\gamma,\gamma',t_0,\vec{t})=\frac{1}{2\pi i}\left(\sum_{-\infty}^{t_0}\text{ or }\sum_{t_0+1}^{+\infty}\right)\psi_j(\gamma,t_0'-1,\vec{t})\psi_j^*(\gamma',t_0',\vec{t}), \tag{2.14}$$

which was obtained for $\vec{t}=0$ earlier in [8].

Lemma 2.2. For $\lambda,\mu\neq\gamma_k,P_0,\infty$ we have:

$$\frac{\partial}{\partial\bar{\lambda}}\omega_j(\lambda,\mu,t_0,\vec{t})=\delta(\lambda-\mu)d\mu d\bar{\mu}, \tag{2.15}$$

where $\delta(\lambda-\mu)$ is the two-dimensional δ-function. For $t_0=0,\vec{t}=0$ we obtain a new representation for the known Cauchy kernel on the Riemann surface (see 1.1).

If the operator $\partial/\partial\bar{\lambda}$ acts on the bundle $\mathbb{B}_j(t_0,\vec{t},D)$ then (2.15) is valid for all λ, μ.

We also use the "vacuum" Cauchy kernel corresponding to $\Gamma=\mathbb{CP}^1,P_0=0$:

$$\omega_j^0(z,z',t_0,\vec{t})=(z/z')^{1-t_0}(z'-z)^{-1}(dz)^j(dz')^{1-j}/2\pi i. \tag{2.16}$$

2.5. *Grassmannians and flag spaces corresponding to j-forms.* Grassmannians in soliton theory and the τ-function were introduced in papers

by the Kyoto group (see [29], [3] and references therein). Here we use the approach by G.Segal and G.Wilson (see [23]) but with an extra discrete parameter t_0 dependence. The tensor properties are not discussed in these works. The Grassmannians of j-differentials and Virasoro action on them were treated in [2]. Now we use the flag spaces instead of Grassmannians. Let S be a small contour in the neighborhood of ∞ such that there are no points of D inside S, and $z = 1/\lambda$ be a local parameter at ∞. Let $H = \mathcal{L}^2(S)$ be the space of square integrable j-forms on S. For each t_0 let us have a decomposition $H = H_+(t_0) \oplus H_-(t_0)$, where $H_+(t_0)$ and $H_-(t_0)$ are subspaces, generated by the basis elements $z^i(dz)^j$ with $i < -t_0$ and $i \geq -t_0$, respectively. The flag space Fl_j is the set of stratified families of subspaces $W = \{W(t_0)\}$ such that $W(t_0) \subset W(t_0+1)$ for all t_0, the orthogonal projections $P_+(t_0) : W(t_0) \to H_+(t_0)$ are Fredholm operators of index 0 and the projections $P_-(t_0) : W(t_0) \to H_-(t_0)$ are compact. Each $W(t_0)$ is an element of the corresponding Grassmannian. Let $w = \{e_k\}$ be a basis in H such that $w(t_0) = \{e_k \mid k < -t_0\}$ is a basis in $W(t_0)$. Let $w_+(t_0) = P_+(t_0)w(t_0), w_-(t_0) = P_-(t_0)w(t_0)$, and $A(t_0) = P_-(t_0)P_+^{-1}(t_0)$. It is convenient to write w as a block matrix whose columns correspond to the Laurent expansions of e_k:

$$w = \begin{bmatrix} w_+ \\ w_- \end{bmatrix}, \; A(t_0) = w_-(t_0)(w_+(t_0))^{-1}. \tag{2.17}$$

The elements $W(t_0)$ which correspond to the given Baker-Akhiezer function $\psi_j(\gamma, t_0, \vec{t})$ consist of all sections $f(\gamma)$ of $\mathbb{B}_j(t_0, \vec{t}, D)$, which are holomorphic in Γ_+. So $\psi_j(\gamma, t_0, \vec{t})$ with different $\vec{t}$ generate $W(t_0)$.

2.6. *Transformations of the flag spaces and τ-function.* Let $W \in \mathrm{Fl}_j$, $W = \{W(t_0)\}$, and g be some linear operator on H from $GL(\infty)$. Then the element $g^{-1}W = \{g^{-1}W(t_0)\} \in \mathrm{Fl}_j$ is the collection of spaces $g^{-1}W(t_0)$ generated by the j-forms $gf(z)$, where $f(z) \in W(t_0)$. Continuous functions act on H by ordinary multiplication, and vector fields on S act on H by Lie differentiation [2]. We also write $W(\vec{t}) = \exp(-\sum z^{-k}t_k)W$.

Remark. There exists some function $\alpha(\gamma)$ such that $\alpha W(t_0) = W(t_0+1)$ for all t_0, but we do not use this fact now.

Let t_0 be a fixed number, $g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a transformation of H (the block form corresponds to the splitting $H = H_+(t_0) \oplus H_-(t_0)$, $a = (g^{-1})_{++}$,

$b = (g^{-1})_{+-}$, $c = (g^{-1})_{-+}$, $d = (g^{-1})_{--})$. Then the τ-function $\tau_W(t_0, g)$ is determined by the formula:

$$\tau_W(t_0, g) = (\det(1 + a^{-1}bA))\tau_W(t_0, 1), \ \tau_j(t_0, \vec{t}) = \tau_W(t_0, \exp(\sum z^{-k}t_k)). \tag{2.18}$$

There is no canonical choice of $\tau_W(1)$ (we omit t_0-dependence in our notations for the sake of brevity), so τ_W is defined up to a constant factor. In [23] $\tau_W(1) = 1$. But if we deform the Riemann surface by a vector field, it is more natural to assume that the variation of $\tau_W(g)$ is given by (2.18). The composition formula for a product of transformations is:

$$\tau_W(g \cdot g_1) = \tau_W(g) \cdot \tau_{(g^{-1}W)}(g_1)/\tau_{(g^{-1}W)}(1) \cdot \rho(g, g_1), \tag{2.19}$$

$$\rho(g, g_1) = \det\{g_{++}(g_1)_{++}(gg_1)^{-1}_{++}\}.$$

We will consider the following transformations of the Grassmannian: the action of the vertex operator in section 2.8 and the action of the vector fields in section 3.3. To calculate the corresponding variations of τ-functions we need the explicit form of $A(t_0) = P_-(t_0)P_+^{-1}(t_0)$.

Remark. Grassmannians as an universal moduli space. Every Riemann surface Γ of finite genus with a bundle and a local parameter generates a point W of the Grassmannian Gr_j (we do not discuss t_0-dependence now and assume $t_0 = 0$). So the Grassmannian can be considered as a universal space including the moduli spaces for all genera. But the points of the Grassmannian corresponding to the Riemann surfaces have the following property (see [12] and references therein). Let $W^\perp \in \mathrm{Gr}_j$ be the set of $1-j$-forms $h(z)(dz)^{1-j}$ such that $\oint f(z)h(z)dz = 0$ for all $f(z)(dz)^j \in W (W^\perp$ corresponds to $\psi^*(\gamma, \vec{t}))$. Then there exists a $1-2j$-form $g(z) = \tilde{g}(z)(dz)^{1-2j}$, $z \in S$, such that $W^\perp = g(z)W$ (see [2]). For an arbitrary $W \in \mathrm{Gr}_j$ this property is not valid. The set of points $W \in \mathrm{Gr}_j$ such that $W^\perp = g(z)W$ for some $g(z)$ is called a universal moduli space.

2.7. *Explicit version of Segal-Wilson construction.* The main operator $A_j(t_0)$ (see 2.17) can be written explicitly via Cauchy kernels (2.13), (2.16): $A_j(t_0)f(\lambda, \vec{0}) = \oint(\omega_j(\lambda, \mu, t_0, \vec{0}) - \omega_j^0(\lambda, \mu, t_0, \vec{0}))f(\mu, \vec{0})$, where $f(\lambda)$ is a j-differential. For transformed Grassmannians $W(\vec{t})$ we have $A_j(t_0, \vec{t}) = \exp(\sum(\lambda^n - \mu^n)t_n)A_j(t_0)$.

2.8. *Vertex operators. Cauchy kernel via τ-function.* Consider the vertex operator $X_j(z) = \exp(\sum z^{-m}t_m - t_0 \ln z) \cdot \exp(-\sum z^n n^{-1}\partial/\partial t_n - \partial/\partial t_0)(dz)^j$

and its adjoint $X_j^*(z) = \exp(t_0 \ln z - \sum z^{-m} t_m) \cdot \exp(\sum z^n n^{-1} \partial/\partial t_n + \partial/\partial t_0)(dz)^{1-j}$ determined near ∞. They satisfy the following commutation relations:

$$X_j(z)X_j^*(u) + X_j^*(u)X_j(z) = -\delta(z-u)du,$$

$$X_j(z)X_j(u) + X_j(u)X_j(z) = 0,$$
$$X_j^*(z)X_j^*(u) + X_j^*(u)X_j^*(z) = 0,$$

so $X_j(z)$ and $X_j^*(z)$ represent fermionic operators on the sphere. Then we have

$$X_j(z)X_j^*(u) \cdot \tau_j(t_0, \vec{t}) = -2\pi i \omega_j(z, u, t_0, \vec{t})\tau_j(t_0, \vec{t}). \tag{2.20}$$

The proof follows from the relation

$$X_j(\lambda)X_j^*(\mu) \cdot \tau_W(g) = \omega_j^0(\lambda, \mu, t_0, \vec{t}) \cdot \tau_W(g(1-k/\lambda)/(1-k/\mu)),$$

where $g = \exp \sum k^m t_m$, from the composition formula (2.19) and the explicit representation 2.7.

2.9. *Algebrogeometrical τ-function for j-forms.* The algebro-geometrical τ-function without t_0-dependence was discussed in [5],[24],[25],[1],[4]. In 2.6 the τ-function was determined up to an arbitrary factor $c(t_0)$. To eliminate this freedom we use the following condition:

$$\tau_j(t_0+1, \vec{t})/\tau_j(t_0, \vec{t}) = \varphi(t_0, \vec{t}), \tag{2.21}$$

φ is defined by (2.12). The equivalent condition for vacuum expectation $< -p \mid p >$ (p being correspondent to t_0 in our paper) was used by I.M.Krichever and S.P.Novikov ((2.34) in [9]). Then we have:

$$\tau_j(t_0+1, \vec{t}) =$$

$$= \exp\left\{\tfrac{1}{2}\left(\sum_1^\infty Q_{ik} t_i t_k + g_2 t_0^2 + 2t_0 \sum_1^\infty (Q_{0k} - q_{0k}) t_k\right) + \sum_1^\infty h_k t_k + g_1 t_0 + g_0\right\} \cdot$$

$$\cdot \theta\left\{(2j-1)\vec{K} + \sum n_k \vec{A}(\gamma_k) + t_0 \vec{U}_0 + \sum_{k\geq 1} t_k \vec{U}_k + \vec{\zeta} \mid B_{mn}\right\}. \tag{2.22}$$

Notations are the same as in 1.2, (2.10). The terms $Q_{0k} - q_{0k}$ and h_k are defined from decompositions $\ln\{(zE(z, P_0)/(E(z, \infty)E(\infty, P_0)dz(\infty))\} = -\sum (Q_{0k} - q_{0k}) z^k/k, \ln \eta = -\sum h_k z^k/k$. The terms g_2, g_1 are determined from (2.21). g_0 is an arbitrary constant for a fixed Riemann surface Γ.

Remark. It is possible to choose the local parameters in ∞ and P_0 so that $g_1 = g_2 = 0$. If $j = 1/2$, then $g_1 \equiv 0$ and one of the local parameters in ∞ or in P_0 may be chosen in an arbitrary way.

The formula (2.22) can be proved from the following connection between the Baker-Akhiezer forms and the τ-function (for $j = 0$ see [3]):

$$\psi_j(\lambda, t_0, \vec{t}) = X_j(\lambda)\tau_j(t_0 + 1, \vec{t})/\tau_j(t_0, \vec{t}),$$

$$\psi_j^*(\lambda, t_0, \vec{t}) = X_j^*(\lambda)\tau_j(t_0 - 1, \vec{t})/\tau_j(t_0, \vec{t}). \tag{2.23}$$

Relation (2.23) follows from 2.7 and (2.18),(2.19).

Chapter 3. Virasoro action on the KP theory objects.

3.1. *The correspondence between nonlinear equations and variations of Riemann surfaces.* Now we calculate the action of the vector fields v on the objects of soliton theory.

Theorem 3.1. Let vector field v on S act on the Riemann surface Γ, and let the divisor $\gamma_1, \ldots, \gamma_g$ and the local parameter $z = 1/\lambda$ be mapped by E. Then the variation of the KP solution χ is given by:

$$\partial\chi(t_0, \vec{t})/\partial\beta = -(2\pi i)^{-1} \oint_S (L_v \psi_j(t_0, \vec{t}))\psi_j^*(t_0, \vec{t}), \tag{3.1}$$

$$(\partial_y - \partial_x^2 - 2\chi_x)\psi_j(t_0, \vec{t}) = 0, \tag{3.2}$$

$$(\partial_y + \partial_x^2 + 2\chi_x)\psi_j^*(t_0, \vec{t}) = 0, \tag{3.3}$$

and the variation of the Baker-Akhiezer function is the following

$$\partial\psi_j(\lambda, t_0, \vec{t})/\partial\beta = (2\pi i)^{-1} \oint_S (L_v \psi_j(\mu, t_0, \vec{t}))\omega_j(\lambda, \mu, t_0, \vec{t}),\ \mu \in S, \tag{3.4}$$

where L_v is the Lie derivative.
The proof follows from (1.12) and asymptotics of ω_j when $\lambda \to \infty$.

Formula (3.4) solves the Krichever-Novikov problem of calculation of the vector field's action on the Baker-Akhiezer function [7].

Remark. (3.1-3) is an integrable Lagrangean system which commutes with the ordinary (commutative) KP hierarchy. The $L - A$ pair for this system is given by (3.2) and (3.4). Using series like (A.3) we obtain a more familiar evolution form. This system is discussed in Appendix A.

Remark. The variation of the wave function ψ_j with the help of kernel (2.13) may be considered as a correct analytical form of infinitesimal Zakharov-Shabat dressing which is valid in the finite-gap case as well as in the decreasing one.

Theorem 3.2. The commutator of flows (3.1-3) as well as (3.4) coincides with the commutator of the vector fields v if the following natural assumptions are valid: 1) The spectral parameter $\lambda = 1/z$ and the divisor D are mapped by E (see 1.4). 2) The asymptotic behaviour of ψ_j and ψ_j^* under the action of v remains fixed: $\psi_j(t_0)\ z^{-to}(dz)^j \cdot \exp \sum z^{-m} t_m$, $\psi_j^*(t_0)\ z^{to-2}(dz)^{1-j} \exp(-\sum z^{-m} t_m)$. 3) We compare Baker-Akhiezer forms on different Riemann surfaces. The connection between the points of different Riemann surfaces is established by the map E. The Baker-Akhiezer functions are compared in these points. 4) In accordance with section 1.4, the vector fields v are assumed to be independent of β functions of z.
The proof follows from direct calculation.

3.2. *Isospectral and non-isospectral symmetries.* Consider the space V of all vector fields on S. It is known that for $g > 1$ V can be presented as a direct sum $V = V_+ \oplus V_0 \oplus V_-$. Here V_+ and V_- are the fields which can be analytically continued to the regions Γ_+ and Γ_-, respectively, and $\dim V_0 = 3g-3 = \dim \mathrm{Ker} \bar{\partial}_2$. The set of Riemann surfaces of genus $g > 1$ can be parameterized by $3g-3$ complex parameters. This set is called a moduli space. There are no natural coordinates on the moduli space but locally we may use $3g-3$ independent elements of Riemann matrix B_{ij}. The action of the vector fields from V_0 at the moduli space is nondegenerate. Therefore, the times of corresponding higher KP equations form local coordinates on the moduli space.

Symmetries corresponding to $v \in V_+, V_-$ do not change the Riemann surface (one can see it from (1.14)), so they are isospectral. Vector fields $v \in V_-$ change the local parameter near ∞. Symmetry action corresponding to $v \in V_+$ comes to the ordinary higher KP symmetry action.

Let us note that for a Riemann surface with two marked points there exists a natural basis of vector fields corresponding to the decomposition $V_+ \oplus V_0 \oplus V_-$- Krichever-Novikov basis [7]. The action of fields $v \in V_+$ on KP theory objects was studied in [7].

3.3. *τ-function variation by complex structure. Variation of* $\det \bar{\partial}_j$. The action of the vector field v on Γ results in the transformation of the flag space

and τ-function (see (2.19)). Using explicit representation 2.5 we obtain

$$\partial \ln \tau_j(t_0,\vec{t})/\partial\beta = (2\pi i)^{-1} \oint_S L_v(\lambda)\omega_j^{\mathrm{reg}}(\lambda,\mu,t_0,\vec{t}) \,|_{\lambda=\mu}, \qquad (3.5)$$

where

$$\omega_j^{\mathrm{reg}}(\lambda,\mu,t_0,\vec{t}) = \omega_j(\lambda,\mu,t_0,\vec{t}) - \omega_j^0(\lambda,\mu,t_0,\vec{t}).$$

The "naive" calculation of variation of $\det\bar\partial_j$ on $\mathbb{B}_j(t_0,\vec{t},D)$ by the complex structure gives

$$\delta\det\bar\partial_j = \det\bar\partial_j \cdot (\det(1+\bar\partial_j^{-1}\kappa\partial_j)-1) = \det\partial_j \cdot \mathrm{Tr}\partial_j\bar\partial_j^{-1}\kappa =$$

$$= \det\bar\partial_j \cdot (2\pi i)^{-1} \oint_S L_v(\lambda)\omega_j^{\mathrm{reg}}(\lambda,\mu,t_0,\vec{t}) \,|_{\lambda=\mu}$$

. We use $\delta\bar\partial_j = \kappa\partial_j$, where κ is the Beltrami differential corresponding to the variations of the complex structure. In our case it is a δ-type function on S. So if we use the same regularization we have

$$\delta\ln\det\bar\partial_j = \delta\ln\tau_j(t_0,\vec{t}).$$

3.4. *Virasoro action on the τ-function. Explicit formulas.* From (2.20), (3.5) it follows that τ-function obeys differential equations:

$$\partial\tau_j(t_0,\vec{t})/\partial\beta_m = L^j\tau_j(t_0,\vec{t}) \qquad (3.6)$$

where the time β_m corresponds to the vector field $v = \lambda^{m+1}d/d\lambda$,

$$L_m^j = \sum(kt_k\partial_{k+m} + \frac{1}{2}(\partial_k\partial_{m-k})) + (t_0 - 2j + (j-\frac{1}{2})(m+1))\partial_m, \; m>0,$$

$$L_0^j = \sum kt_k\partial_k + \frac{1}{2}(t_0-2j)^2 + (j-1/2)(t_0-2j), \qquad (3.7)$$

$$L_m^j = \sum(kt_k\partial_{k+m} + \frac{1}{2}k(m-k)t_{-k}t_{k-m}) - m(t_0-2j+(j-\frac{1}{2})(m+1))t_{-m}, \; m<0.$$

Here $\partial_k = \partial/\partial t_k$ and indices are assumed to be positive. Operators L_m^j form the Virasoro algebra with central charge $c_j = 6j^2-6j+1$.

Remark. Using the expression for KP solution $\chi(t_0,\vec{t}) = \partial\ln\tau_j(t_0,\vec{t})/\partial t_1$ we obtain the formula for the variation of χ derived in the different way described in 3.1.

Remark. Substituting (2.22) into (3.6) we obtain variations of B_{mn}, Q_{ik}, $\vec{U}_k$, $\vec{\zeta}$, h_k, g_2, g_1, g_0 and other geometrical objects on Γ by varying the complex structure (see Appendix B).

Appendix A. Higher KP symmetries. The ordinary higher KP equations corresponding to the times t_m are the symmetries of the KP itself (i.e., they commute with it). They mutually commute and do not explicitly depend upon t_m. These symmetries are a part of a broader hierarchy parameterized by two integers m, n (see [10] and references therein). These equations explicitly depend upon t_m and commute with the ordinary KP-hierarchy, but in general they do not commute with each other. They are

$$\partial\chi/\partial\beta_{mn} = \mathrm{res}_{\lambda=\infty}\,(\lambda^m((\partial/\partial\lambda)^n w(\lambda,\vec{t}))w^*(\lambda,\vec{t})), \tag{A.1}$$

where $w(\lambda, t)$ and $w^*(\lambda,\vec{t})$ satisfy the auxiliary linear problem

$$(\partial_y - \partial_x^2 - 2\chi_x)w(\lambda,\vec{t}) = 0,\ (\partial_y + \partial_x^2 + 2\chi_x)w^*(\lambda,\vec{t}) = 0 \tag{A.2}$$

and have asymptotic behaviour

$$\begin{gathered} w(\lambda,\vec{t}) = (1 + \textstyle\sum w_n(\vec{t})\lambda^{-n}))\exp\textstyle\sum t_n\lambda^n, \\ w^*(\lambda,\vec{t}) = (1 + \textstyle\sum w_n^*(\vec{t})\lambda^{-n})\exp(-\textstyle\sum t_n\lambda^n), \end{gathered} \tag{A.3}$$

when $\lambda \to \infty$; β_{mn} is the corresponding time. Equations (A1)-(A2) can be written in the simple Lagrangean form, w_n, w_n^* and χ being independent variables. Expressing recurrently w_n, w_n^* via χ from (A.2) and substituting them into (A.1), we obtain a more familiar form of higher KP equations which are nonlocal evolution equations at one function χ. For $n = 0$ we have the ordinary (commutative) KP-hierarchy, β_{mo} being equal to t_m. When $n = 1$ we obtain conformal symmetries. Tensor properties were not treated in [10].

Symmetries (A.1) admit another description, one like the description in [3]. Let $K = 1 + \sum_1^\infty K_n\partial^{-n}$, where $\partial = \partial/\partial x$, be a pseudodifferential operator and let $L = K\circ\partial\circ K^{-1}$, $M = K\circ(\sum_1^\infty mt_m\partial^{m-1})\circ K^{-1}$, $[L, M] = 1$, where $\circ$ and $[,]$ denote, respectively, the product and the commutator in the algebra of the pseudodifferential operators [27]. Let $()_-$ be the projector $(\sum f_n\partial^n)_- = \sum_{n<0} f_n\partial^n$. Consider equation

$$\partial L/\partial\beta_{nm} = [L, (M^n \circ L^m)_-] = 0. \tag{A.4}$$

It is compatible with the ordinary KP hierarchy $\partial L/\partial t_k = [L, (L^k)_-]$. For $\chi = K_1$ one can obtain (A.1).

It was noted [10] that invariant solutions for these symmetries can be described in terms of the isomonodromy problem [5], [30].

Appendix B. The variations of geometrical objects. From (2.22) and (3.5) we have for $j = 1/2$, $n > 0$:

$$\partial g_0/\partial\beta_n = \sum_1^{n-1} Q_{m,n-m}/2, \tag{B.1}$$

$$\partial Q_{mk}/\partial\beta_n = mQ_{n+m,k} + kQ_{m,n+k} + \sum_1^{n-1} Q_{l,m}Q_{n-l,k}, \tag{B.2}$$

$$\partial\vec{\zeta}/\partial\beta_n = 0, \tag{B.3}$$

$$\partial\vec{U}_k/\partial\beta_n = k\vec{U}_{n+k} + \sum_1^{n-1} \vec{U}_m Q_{n-m,k}, \tag{B.4}$$

$$\partial\vec{U}_0/\partial\beta_n = \vec{U}_n + \sum_1^{n-1}(Q_{0,n-m} - q_{0,n-m})\vec{U}_m, \tag{B.5}$$

$$\partial(Q_{0,k} - q_{0,k})/\partial\beta_n = Q_{kn} + k(Q_{0,k+n} - q_{0,k+n}) + \sum(Q_{0,l} - q_{0,l})Q_{k,n-l}, \tag{B.6}$$

$$\partial g_2/\partial\beta_n = \sum(Q_{0,k} - q_{0,k})(Q_{0,n-k} - q_{0,n-k}) + 2(Q_{0,n} - q_{0,n}), \tag{B.7}$$

$$\partial B_{kl}/\partial\beta_n = 2\pi i\sum_1^{n-1}(\vec{U}_m)_k(\vec{U}_{n-m})_l. \tag{B.8}$$

For $j = 1/2$, $n = 0$:

$$\partial Q_{mn}/\partial\beta_0 = (m+n)Q_{mn}, \tag{B.9}$$

$$\partial\vec{U}_k/\partial\beta_0 = k\vec{U}_k, \tag{B.10}$$

$$\partial(Q_{0,k} - q_{0,k})/\partial\beta_0 = k(Q_{0,k} - q_{0,k}), \tag{B.11}$$

$$\partial g_2/\partial\beta_0 = 1. \tag{B.12}$$

The other derivatives are equal to zero. For $j = 1/2$, $n < 0$ we obtain:

$$\partial Q_{kl}/\partial\beta_n = kl\delta_{k+n+l,0} + k\vartheta(k+n)Q_{n+k,l} + l\vartheta(l+n)Q_{k,n+l} \tag{B.13}$$

$$\partial\vec{U}_k/\partial\beta_n = k\vec{U}_{n+k}\vartheta(k+n) \tag{B.14}$$

$$\partial(Q_{0,k}-q_{0,k})/\partial\beta_n = k\delta_{k+n,0} + k(Q_{0,k+n}-q_{0,k+n})\vartheta(k+n) \qquad (B.15)$$

The other derivatives are equal to zero. Here $\vartheta(k)=1$ if $n>0$, or $=0$ if $n\le 0$.

Appendix C. Krichever-Novikov fermions. Let b_k, c_k be fermionic operators with the usual anticommutators: $[b_n,b_m]_+=0$, $[c_n,c_m]_+=0$, $[c_n,b_m]=\delta_{nm}$, and $\mid 0>(<0\mid)$ be right (left) vacuum vectors with the properties:

$$\begin{array}{ll} b_n \mid 0>=0 \quad (n\ge 0), & c_n\mid 0>=0 \quad (n<0) \\ <0\mid b_n=0 \quad (n<0), & <0\mid c_n=0 \quad (n\ge 0) \end{array} \qquad (C.1)$$

Put $\mid k>=C_k\mid 0>$, $<k\mid=<0\mid B_k$ where $\mid k>(<k\mid)$ denotes states with the "charge" k $(-k)$ and

$$B_k=\begin{cases} c_{-1}\cdots c_k & (k<0)\\ 1 & (k=0)\\ b_0\cdots b_{k-1} & (k>0)\end{cases}, \quad C_k=\begin{cases} b_k\cdots b_{-1} & (k<0)\\ 1 & (k=0)\\ c_{k-1}\cdots c_0 & (k>0)\end{cases}. \qquad (C.2)$$

Let us introduce the following fermion operators on the Riemann surface Γ by analogy with [8],[9]:

$$b(\gamma)=\sum b_n\psi_j(\gamma,n,\vec{t}), \quad c(\gamma)=\sum c_n\psi_j^*(\gamma,n+1,\vec{t}), \quad \gamma\in\Gamma,$$

which are j- and $1-j$-forms on Γ and $n=t_0$ (see 2.3). Now all correlation functions are expressed in terms of Baker-Akhiezer functions. From (2.14) it follows:

$$<t_0\mid b(\gamma')c(\gamma)\mid t_0>=\omega_j(\gamma',\gamma,t_0,\vec{t}), \qquad (C.3)$$

where $\omega_j(\gamma',\gamma,t_0,\vec{t})$ is the Cauchy-Baker-Akhiezer kernel (see 2.4).

The Baker-Akhiezer functions can be expressed via fermions in a way similiar to [3]:

$$\psi_j(\gamma,n,\vec{t})=<n\mid b(\gamma,\vec{t})\mid n+1>, \psi_j^*(\gamma,n+1,\vec{t})=<n+1\mid c(\gamma,\vec{t})\mid n>. \qquad (C.4)$$

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